ANALYSIS 2017.

1. All-around

Problem 1.1. Let $f : \mathbb{D} \to \mathbb{D}$ be a holomorphic function, where $\mathbb{D}$ is the unit disk. If $f(0) = 0$, then

$$|z|\frac{|f'(0)| - |z|}{1 - |f'(0)||z|} \le |f(z)| \le |z|\frac{|f'(0)| + |z|}{1 + |f'(0)||z|}$$

Problem 1.2. Suppose $f(x)$ is a convex function defined on the whole real axis with the condition that

$$|f(x)| \le C(1 + |x|)$$

for some constant C. Prove

(1) $\lim_{x\to+\infty} \frac{f(x)}{x} = \alpha$ exists for some real number α.
(2) There is a β such that $f(x) \le \alpha x + \beta$ for x large.
(3) Is there a constant γ such that

$$f(x) \ge \alpha x + \gamma$$

for all x?

Problem 1.3. Can you solve the equation

$$\begin{cases} \Delta u = \frac{x}{x^2+y^2}, & \text{in } \{(x, y) : x^2 + y^2 < 1\}, \\ u = 0, & \text{on } \{(x, y) : x^2 + y^2 = 1\} \end{cases}$$

in the weak sense? What is the optimal continuity property of u?